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STRICTLY CONVEX METRIC SPACES AND FIXED POINTS

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Abstract. In this article we examine strictly convex metric space and strictly convex metric space with convex round balls. These objects generalize well known concept of strictly convex Banach space. We prove fixed point theorems for nonexpansive, quasi-nonexpansive and asymptotically nonexpansive mappings in strictly convex metric space with convex round balls. These results extend previous results of R. de Marr, F. E. Browder, W. A. Kirk, K. Goebel, W. G. Dotson, T. C. Lim and some others.

1. Introduction

We can define convexity in the ordinary sense only in vector space. In the bibliography we find several possibilities as concept of convexity of vector space transfer to space with metric or topology. What properties of convexity are essential? From works K. Menger [20], T. Botts [2], W. L. Klee [15], D. C. Kay and E.W. Womble [11], M. van de Vel [25] solid indications are two:

1. intersection of a convex sets is convex set;
2. closed balls are convex sets.

If we can guarantee these properties in considered space, we say structure of convexity is formed in space.

The condition of convexity for definition set or range set of mapping very often is used for existence of fixed point of mapping in the theory of fixed points. Several mathematicians have attempted transfer of structure of convexity to space which is not vector space. For example, to metric space - W. Takahashi [23], J. P. Penot [22], W. A. Kirk [13], [14], to topological

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space - M. R. Tasković [24] and to freely set (with help of closure operators) - A. Liepiņš [17].

In this article we study strictly convex metric space and strictly convex metric space with convex round balls and generalize concept of strictly convex Banach space. Some interesting results in fixed point theory has been proved in strictly convex Banach space, for example, M. Edelstein [6], [7], F. E. Browder [4] (uniformly convex Banach space is also strictly convex), L. P. Belluce and W. A. Kirk [1], Z. Opial [21], W. G. Dotson [5], K. Goebel and W. A. Kirk [9], P. Kuhfitting [16]. We prove some fixed point theorems for nonexpansive mapping and for commutative families of nonexpansive or quasi-nonexpansive or asymptotically nonexpansive mappings in strictly convex metric space with convex round balls.

2. Basic concepts

We know:

Definition 2.1. A Banach space X is said to be strictly convex if all points of unit sphere are not inner points of straight lines in a unit ball.

V. I. Istrăţescu [10] has proved that in Banach space X the following conditions are equivalent:

1. X is a strictly convex space;
2. $\forall x, y \in B(0,1),\ x \neq y: \quad \| x+y \| < 2$ (with $B(a,r)$ we denote closed ball with center a and radius r);
3. $\forall x, y \in X: \quad \| x+y \| = \| x \| + \| y \| \Longrightarrow$
   $\Longrightarrow ((\exists \lambda > 0 : x = \lambda y) \bigvee (x = 0) \bigvee (y = 0))$.

It is known that in convex closed subset of a strictly convex Banach space set of fixed points for nonexpansive mapping is convex and closed. We note that:

Definition 2.2. A mapping $f : X \to X$, where X is a metric space, is said to be nonexpansive if for every $x, y \in X$ inequality $d(f(x), f(y)) \leq d(x,y)$ holds.

This property is also true for broader classes of mappings, for example, for quasi-nonexpansive and asymptotically nonexpansive mappings.

Definition 2.3 (V. G. Dotson [5]). A self-mapping f of a subset K of a normed linear space is said to be quasi-nonexpansive provided f has at least one fixed point in K, and if $p \in K$ is any fixed point of f then

$$\| f(x) - p \| \leq \| x - p \|$$

holds for all $x \in K$.

Definition 2.4 (K. Goebel and W. A. Kirk [9]). A self-mapping f of a subset K of a normed linear space is said to be asymptotically nonexpansive if for each paar $x, y \in K$:

$$\| f^i(x) - f^i(y) \| \leq k_i \| x - y \|,$$

where $(k_i)_{i \in \mathbf{N}}$ is a sequence of real numbers such that $\lim_{i \to \infty} k_i = 1$ (it is assumed that $k_i \geq 1$ and $k_i \geq k_{i+1}$, $i = 1, 2, \dots$).

Let (X, d) be a metric space with distance d.

Definition 2.5. A set $K \subset X$ is said to be convex if for each $x, y \in K$ and for each $t \in [0; 1]$ there exists $z \in K$ that satisfies:

$$d(x, z) = td(x, y) \text{ and } d(z, y) = (1 - t)d(x, y).$$

We note that by means of this Definition 2.5 closed balls may be non-convex sets and intersection of a convex sets may be non-convex set (see, for example, I. Galiņa [8]). Therefore we define strictly convex metric space in following manner:

Definition 2.6. A metric space X is said to be strictly convex if for each $x, y \in X$ and for each $t \in [0; 1]$ there exists *unique* $z \in X$ that satisfies:

$$d(x, z) = td(x, y) \text{ and } d(z, y) = (1 - t)d(x, y).$$

This is not new original definition; we can find in bibliography, for example, in W. Takahashi [23]. But we can not find comparison with strictly convex Banach space. The author of this article has proved in 1992 (I. Galiņa [8]): the following conditions are equivalent in Banach space X:

1. $\forall x, y \in X: \quad \| x + y \| = \| x \| + \| y \| \Longrightarrow$
   $\Longrightarrow ((\exists \lambda > 0 : x = \lambda y) \bigvee (x = 0) \bigvee (y = 0));$
2. $\forall x, y \in X \quad \forall t \in [0; 1] \quad \exists! z \in X:$
   $\| x - z \| = t \| x - y \|, \quad \| z - y \| = (1 - t) \| x - y \|.$

Since the first condition is equivalent with concept of strictly convex Banach space, we conclude that strictly convex Banach space indeed is strictly convex metric space in particular case.

It is simple to prove that intersection of convex sets (in means of Definition 2.5) is convex set in strictly convex metric space (I. Galiņa [8]). But question of convexity of closed balls is still open.

3. Strictly convex metric space with convex round balls

Since we can not guarantee that closed balls in strictly convex metric space are convex sets, we require this condition in addition. We define:

Definition 3.1. A strictly convex metric space X is said to be strictly convex space with convex round balls if

$$\forall a, b, c \in X (a \neq b) \forall t \in]0; 1[\exists z \in X :$$
$$d(a, z) = td(a, b) \quad \text{and} \quad d(z, b) = (1 - t)d(a, b),$$

$$d(c, z) < \max\{d(c, a), d(c, b)\}. \tag{3.1}$$

Lemma 3.1. *Let X be a strictly convex metric space with convex round balls. Then closed ball $B(c, r) := \{y \in X \mid d(c, y) \leq r\}$ for every $r > 0$ and every $c \in X$ is a convex set.*

Proof. We fix $r > 0$ and $c \in X$. We choose freely two points $a, b \in B(c, r)$, $a \neq b$, and fix $t \in]0; 1[$. By definition of strictly convex metric space, there exists unique $z \in X$ such that $d(a, z) = td(a, b)$ and $d(z, b) = (1 - t)d(a, b)$. We must prove that $z \in B(c, r)$ or $d(c, z) \leq r$.

If $t \in]0; 1[$ then $d(a, z) = td(a, b)$, $d(z, b) = (1-t)d(a, b)$ and by condition of convex round balls follows that

$$d(c, z) < \max\{d(c, a), d(c, b)\} \leq r. \quad \square$$

It can be proved that condition

$$\forall a, b, c \in X (a \neq b) \quad \forall t \in]0; 1[\quad \exists z \in X :$$
$$d(a, z) = td(a, b) \quad \text{and} \quad d(z, b) = (1 - t)d(a, b)$$
$$\text{and } d(c, z) \leq \max\{d(c, a), d(c, b)\}$$

is equivalent with condition of convexity of closed balls. We require more. With help of strict inequality (3.1) we can prove Lemmas 3.2 and 3.3. Besides this strict inequality shows that if a and b belongs to sphere of ball $B(c, r)$ then z does not belong to this sphere, i.e., sphere does not contain straight lines therefore in Definition 3.1 we speak of convex round balls.

We notice that well known metric space $\mathbf{R}$ with module metric and $\mathbf{R}^2$ with Euclidean metric is both strictly convex metric space and strictly convex metric space with convex round balls. But $\mathbf{R}^2$ with maximum metric is not strictly convex metric space. Trivial example for strictly convex metric space that is not strictly convex metric space with convex round balls is space with one point x and $d(x, x) = 0$. We notice that every convex subset of strictly convex Banach space is strictly convex metric space but no more strictly convex Banach space.

Strictly convex metric spaces with convex round balls inherent some good properties that we formulate as lemmas.

Lemma 3.2. *Let X be a strictly convex metric space with convex round balls and K be a convex and compact subset of X and $y \in X$. Then*

$$\exists! y_0 \in K: \quad d(y, y_0) = \inf\{d(x, y) \mid x \in K\}.$$

Proof. We define in set K functional f with identity $f(x) := d(x, y)$, $\forall x \in K$. Since K is a compact set then by Weierstrass theorem: $\exists y_0 \in K$:

$$f(y_0) = \inf\{f(x) \mid x \in K\} \quad \text{or} \quad d(y, y_0) = \inf\{d(x, y) \mid x \in K\}.$$

The uniqueness we prove from contrary. We suppose that there exists another point $y_0' \in K$ such that $d(y, y_0') = \inf\{d(x, y) \mid x \in K\}$. The set K is convex therefore for fixed $t \in]0;1[$ exists such $y_0'' \in K$ that

$$d(y_0, y_0'') = td(y_0, y_0') \quad \text{and} \quad d(y_0'', y_0') = (1 - t)d(y_0, y_0').$$

From condition of convex round balls follows that

$$d(y, y_0'') < \max\{d(y, y_0), d(y, y_0')\} = \inf\{d(x, y) \mid x \in K\}.$$

We have obtained that point y_0'' to be closer than points y_0 and y_0'. The contradiction completes the proof. □

There is a certain course in fixed point theory formed by selfmappings of sets with normal structure. This concept has been worked out by M. Brodskij and D. Milman [3] in 1948.

Definition 3.2. A convex set K in a metric space X is said to have normal structure if for each bounded and convex subset $H \subset K$, that contains more than one point, there is some point $y \in H$ such that

$$\sup\{d(x, y) \mid x \in H\} < \operatorname{diam} H = \sup\{d(x, y) \mid x, y \in H\}.$$

We can prove:

Lemma 3.3. *Every convex and compact set in strictly convex metric space X with convex round balls has normal structure.*

Proof. Suppose K is convex and compact set in space X that does not have normal structure. Then exists a compact and convex subset $H \subset K$ that contains more than one point and

$$\forall x \in H: \quad \sup\{d(x, y) \mid y \in H\} = \operatorname{diam} H.$$

We choose point $x_1 \in H$. Then $\exists x_2 \in H$: $d(x_1, x_2) = \operatorname{diam} H$. Since H is convex set then for fixed $t \in]0;1[$ exists $z \in H$ such that $d(x_1, z) = td(x_1, x_2)$ and $d(z, x_2) = (1 - t)d(x_1, x_2)$. Since $z \in H$ then $\exists x_3 \in H$ that $d(x_3, z) = \operatorname{diam} H$. But then by condition of convex round balls:

$$d(x_3, z) = \operatorname{diam} H < \max\{d(x_1, x_3), d(x_2, x_3)\} \leq \operatorname{diam} H.$$

The contradiction completes the proof. □

4. Fixed points

In addition to classic case we can prove that set of fixed points for nonexpansive selfmappings in convex closed subset of strictly convex metric space is convex and closed (I. Galiņa [8]).

If we replace in Definitions 2.3 and 2.4 the vector space with metric space and norm with metric then two followings lemmas are true.

Lemma 4.1. *Let K be a convex and closed subset of strictly convex metric space X. If mapping $f : K \to K$ is a quasi-nonexpansive then the set of all fixed points of mapping f $Fixf$ is closed and convex.*

Proof. Since f is quasi-nonexpansive then $Fixf \neq \emptyset$ and f is continuous mapping in all fixed points. We assume, that $Fixf$ is not closed set. Then exists x that belongs to boundary of $Fixf$ and that does not belong to $Fixf$. Since K is closed set then $x \in K$. Since $x \notin Fixf$ then $f(x) \neq x$. We define $r := \frac{1}{3}d(f(x), x) > 0$. Then exists $y \in Fixf$ that $d(x, y) \leq r$. Since f is quasi-nonexpansive then $d(f(x), y) \leq d(x, y) \leq r$, and we have:

$$3r = d(f(x), x) \leq d(f(x), y) + d(y, x) \leq 2r.$$

This contradiction shows that assumption of $Fixf$ un-closedness is false.

Now we prove that $Fixf$ is convex set. We choose freely two points x and y $(x \neq y)$ in set $Fixf$. Let $t \in]0; 1[$. We find the corresponding $z \in K$: $d(x, z) = td(x, y)$ and $d(z, y) = (1 - t)d(x, y)$, which is unique by strictly convexity of X. We want to prove, that point z belongs to set $Fixf$. Since f is quasi-nonexpansive then

$$d(f(z), x) \leq d(z, x) \quad \text{and} \quad d(f(z), y) \leq d(z, y).$$

Therefore

$$\begin{aligned} d(x, y) \leq d(x, f(z)) + d(f(z), y) \leq d(z, x) + d(z, y) = \\ = td(x, y) + (1 - t)d(x, y) = d(x, y). \end{aligned}$$

It follows that

$$d(x, f(z) = d(z, x) = td(x, y), \qquad d(f(z), y) = d(z, y) = (1 - t)d(x, y).$$

By strictly convexity of X implies that $z = f(z)$ and $z \in Fixf$, i.e., $Fixf$ is convex set. □

Lemma 4.2. *Let K be convex and closed subset of strictly convex metric space X. If mapping $f : K \to K$ is an asymptotically nonexpansive then the set of all fixed points of mapping f $Fixf$ is closed and convex.*

Proof. From continuity of mapping f follows closedness of set $Fixf$. We prove that $Fixf$ is convex set.
We choose freely two points x and y ($x \neq y$) in $Fixf$, then

$$f^i(x),\ f^i(y) \in Fixf, \quad i = 1, 2, \dots .$$

Let $t \in]0;1[$. We find the corresponding

$$z \in K: \quad d(x,z) = td(x,y),\ d(z,y) = (1-t)d(x,y). \tag{4.1}$$

Sine X is strictly convex then z is unique. We will have to prove that $z \in Fixf$ or $z = f(z)$.

From definition of asymptotically nonexpansive mapping follows that:

$$d(f^i(z),x) = d(f^i(z), f^i(x)) \leq k_i d(z,x) = tk_i d(x,y), \quad i = 1, 2, \dots \tag{4.2}$$

$$d(f^i(z),y) = d(f^i(z), f^i(y)) \leq k_i d(z,y) = (1-t)k_i d(x,y), \quad i = 1, 2, \dots \tag{4.3}$$

Inequality of triangle and (4.2) and (4.3) implies:

$$d(x,y) \leq d(x, f^i(z)) + d(f^i(z), y) \leq$$
$$\leq tk_i d(x,y) + (1-t)k_i d(x,y) = d(x,y),\ i = 1, 2, \dots .$$

Let i tend to infinity. Then $\lim_{i\to\infty} k_i = 1$ and

$$d(x, \lim_{i\to\infty} f^i(z)) + d(\lim_{i\to\infty} f^i(z), y) = td(x,y) + (1-t)d(x,y).$$

From (4.2) and (4.3) follows that

$$d(x, \lim_{i\to\infty} f^i(z)) = td(x,y),$$
$$d(\lim_{i\to\infty} f^i(z), y) = (1-t)d(x,y).$$

z is a unique point with property (4.1) therefore $\lim_{i\to\infty} f^i(z) = z$. It follows that

$$z = \lim_{i\to\infty} f^i(z) = \lim_{i\to\infty} f^{i+1}(z) = f(\lim_{i\to\infty} f^i(z)) = f(z),$$

i.e., $z \in Fixf$ and $Fixf$ is convex set. □

Inspired from fixed point theorems where condition of normal structure is used (for example, R. de Marr [19], W. A. Kirk [12], W. Takahashi [23] or M. R. Tasković [24]) we can prove:

Theorem 4.1. *Let X be strictly convex metric space with convex round balls. Let $K \subset X$ be convex and compact set. If $f: K \to K$ is nonexpansive mapping then f has a fixed point in K.*

Proof. From Zorn's lemma, minimal element K_0 exists in the collection of all nonempty convex and closed subsets of K, each of them is mapped into itself by f. We show that K_0 consists of a single point. We assume that $\text{diam}K_0 > 0$.

Since K_0 is convex set then by Lemma 3.3 K_0 has normal structure, i.e.,

$$\exists x \in K_0 : \sup\{d(x,y) \mid y \in K_0\} = r < \text{diam}K_0.$$

We denote convex closed hull of set $f(K_0)$ with $\overline{co}f(K_0) = K_1$. Since $f(K_0) \subset K_0$ then

$$K_1 = \overline{co}f(K_0) \subset \overline{co}K_0 = K_0 \quad \text{and}$$
$$f(K_1) \subset f(K_0) \subset \overline{co}f(K_0) = K_1.$$

The minimality of K_0 implies $K_1 = K_0$.

We define set

$$C := (\cap_{y \in K_0} B(y,r)) \cap K_0.$$

That is nonempty since $x \in C$, that is convex (by Lemma 3.1 balls are convex sets) and closed set as intersection of convex and closed sets.

We define set

$$C_1 := \left(\cap_{y \in f(K_0)} B(y,r)\right) \cap K_0.$$

Since $f(K_0) \subset K_0$ then $C_1 \supset C$. If $z \in C_1$ then

$$f(K_0) \subset B(z,r) \quad \text{and } K_0 = K_1 = \overline{co}f(K_0) \subset B(z,r)$$

(because $B(z,r)$ is closed and convex set) therefore $C \supset C_1$. It follows that $C = C_1$.

We choose $z \in C$ and $y \in f(K_0)$. Then exists $x \in K_0$ such that $y = f(x)$. Thereby:

$$d(f(z),y) = d(f(z),f(x)) \leq d(z,x) \leq r,$$

i.e., $f(z) \in C_1$. Since $C = C_1$ then $f(z) \in C$ or $f(C) \subset C$. The minimality of K_0 implies $C = K_0$. But

$$\text{diam}C \leq r < \text{diam}K_0.$$

From obtained contradiction we conclude that $\text{diam}K_0 = 0$ and $K_0 = \{x^*\}$ and therefore $f(x^*) = x^*$. □

We generalize Theorem 4.1 for commutative family of nonexpansive mappings.

Definition 4.1. A family of mappings F is commutative if for all $x \in K$, where K is an arbitrary set, condition $f(g(x)) = g(f(x))$ holds for all $f, g \in F$.

Our result generalize fixed point theorems for commutative family of nonexpansive mappings of R. de Marr [19], F. E. Browder [4] and T. C. Lim [18].

Theorem 4.2. *Let X be a strictly convex metric space with convex round balls. Let $K \subset X$ is convex and compact set. If $F = \{f \mid f : K \to K\}$ is commutative family of nonexpansive mappings then exists a common fixed point for family F, i.e.,*

$$\exists x^* \in K \quad \forall f \in F: \quad f(x^*) = x^*.$$

Proof. From Theorem 4.1 it is known that $Fixf \neq \emptyset$, $\forall f \in F$. Since X is strictly convex metric space, $Fixf$ is convex and closed sets for every $f \in F$ (by I. Galiņa [8]).

Let us inductively prove that $\cap_{i=1}^{n} Fixf_i \neq \emptyset$ for every $n \in \mathbf{N}$. For $n = 1$ the statement is true from the Theorem 4.1. Assuming that $\cap_{i=1}^{k} Fixf_i \neq \emptyset$, let us prove that $\cap_{i=1}^{k+1} Fixf_i \neq \emptyset$. Since by assumption

$$f_{k+1}(x) = f_{k+1}(f_i(x)) = f_i(f_{k+1}(x)), \quad i = 1, 2, \ldots, k,$$

it follows that

$$f_{k+1}(x) \in \cap_{i=1}^{k} Fixf_i \quad \text{and hence}$$
$$f_{k+1} : \cap_{i=1}^{k} Fixf_i \to \cap_{i=1}^{k} Fixf_i.$$

Let us prove that the mapping f_{k+1} has a fixed point in the set $\cap_{i=1}^{k} Fixf_i$. The sets $Fixf_i$, $i = 1, 2, \ldots, k$, are nonempty, closed and convex, therefore $\cap_{i=1}^{k} Fixf_i$ is closed and convex as intersection of closed and convex sets; that is compact set as closed subset of compact set K. By Theorem 4.1 for nonexpansive mapping there exists

$$f_{k+1} : \cap_{i=1}^{k} Fixf_i \to \cap_{i=1}^{k} Fixf_i$$

fixed point in set $\cap_{i=1}^{k} Fixf_i$, therefore

$$\cap_{i=1}^{k+1} Fixf_i \neq \emptyset.$$

Since set K is compact then $\cap_{f \in F} Fixf$ is nonempty set also for infinite family of mappings. □

Similar theorems we can to prove for commutative family of quasi-nonexpansive and asymptotically nonexpansive mappings.

Theorem 4.3. *Let X be strictly convex metric space with convex round balls. Let $K \subset X$ be convex and compact set. If $F = \{f \mid f : K \to K\}$ is commutative family of quasi-nonexpansive mappings then exists a common fixed point for family F.*

Proof. Idea of proof is similar as in Theorem 4.2. The differences are that for $n = 1$ the statement is true by definition of quasi-nonexpansive mapping and the proof that mapping f_{k+1} has a fixed point in the set $\cap_{i=1}^{k} Fixf_i$.

Let

$$f_{k+1} : \cap_{i=1}^{k} Fixf_i \to \cap_{i=1}^{k} Fixf_i.$$

Let $\cap_{i=1}^{k} Fixf_i$ be convex and compact set (this follows from Lemma 4.1). We fix $z \in Fixf_{k+1} \neq \emptyset$. By Lemma 3.2 there exists unique element

$$z_0 \in \cap_{i=1}^{k} Fixf_i \quad \text{such that } d(z, z_0) = \inf\{d(z, y) \mid y \in \cap_{i=1}^{k} Fixf_i\}.$$

Then from definition of quasi-nonexpansive mapping we get:

$$d(z, z_0) = \inf\{d(z, y) \mid y \in \cap_{i=1}^{k} Fixf_i\} \leq$$
$$\leq d(z, f_{k+1}(z_0)) = d(f_{k+1}(z), f_{k+1}(z_0)) \leq d(z, z_0).$$

From uniqueness of z_0 follows that $f_{k+1}(z_0) = z_0$ therefore

$$z_0 \in \cap_{i=1}^{k+1} Fixf_i \neq \emptyset. \quad \square$$

Theorem 4.4. *Let X be strictly convex metric space with convex round balls. Let $K \subset X$ is convex and compact set. If $F = \{f \mid f : K \to K\}$ is commutative family of asymptotically nonexpansive mappings and*

$$\forall f \in F : \ Fixf \neq \emptyset$$

then exists a common fixed point for family F.

Proof. Proof is similar to previous theorems. The differences are following. The set $\cap_{i=1}^{k} Fixf_i$ is convex and closed by Lemma 4.2, since K is convex and compact set then $\cap_{i=1}^{k} Fixf_i$ is convex and compact set. We prove that mapping

$$f_{k+1} : \cap_{i=1}^{k} Fixf_i \to \cap_{i=1}^{k} Fixf_i$$

has a fixed point in set $\cap_{i=1}^{k} Fixf_i$. We fix $z \in Fixf_{k+1} \neq \emptyset$. By Lemma 3.2 there exists unique element $z_0 \in \cap_{i=1}^{k} Fixf_i$ such that

$$d(z, z_0) = \inf\{d(z, y) \mid y \in \cap_{i=1}^{k} Fixf_i\}.$$

Then:

$$d(z, z_0) = \inf\{d(z, y) \mid y \in \cap_{i=1}^{k} Fixf_i\} \leq$$
$$\leq d(z, f_{k+1}^{i}(z_0) = d(f_{k+1}^{i}(z), f_{k+1}^{i}(z_0) \leq k_i d(z, z_0), \ i = 1, 2, \ldots .$$

Let $i \to \infty$. Then

$$\lim_{i \to \infty} k_i = 1 \quad \text{and } d(z, z_0) = d(z, \lim_{i \to \infty} f_{k+1}^{i}(z_0)).$$

From uniqueness of z_0 follows that

$$z_0 = \lim_{i \to \infty} f_{k+1}^{i}(z_0).$$

Since

$$z_0 = \lim_{i \to \infty} f_{k+1}^{i}(z_0) = \lim_{i \to \infty} f_{k+1}^{i+1}(z_0) = f(\lim_{i \to \infty} f_{k+1}^{i}(z_0)) = f(z_0)$$

then

$$z_0 \in \cap_{i=1}^{k+1} Fixf_i \neq \emptyset. \quad \square$$

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